LIMITS OF TRANSLATES OF DIVERGENT GEODESICS AND INTEGRAL POINTS ON ONE-SHEETED HYPERBOLOIDS

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Abstract. For any non-uniform lattice Γ in $\mathrm{SL}_2(\mathbb{R})$, we describe the limit distribution of orthogonal translates of a *divergent* geodesic in $\Gamma\backslash \mathrm{SL}_2(\mathbb{R})$. As an application, for a quadratic form Q of signature $(2,1)$, a lattice Γ in its isometry group, and $v_0 \in \mathbb{R}^3$ with $Q(v_0) > 0$, we compute the asymptotic (with a logarithmic error term) of the number of points in a discrete orbit $v_0\Gamma$ of norm at most T, when the stabilizer of v_0 in Γ is finite. Our result in particular implies that for any non-zero integer d, the smoothed count for number of integral binary quadratic forms with discriminant d^2 and with coefficients bounded by T is asymptotic to $c \cdot T \log T + O(T)$.

1. Introduction

1.1. **Motivation.** Let $Q \in \mathbb{Z}[x_1, \cdots, x_n]$ be a homogeneous polynomial and set $V_m := \{x \in \mathbb{R}^n : Q(x) = m\}$ for an integer m. It is a fundamental problem to understand the set $V_m(\mathbb{Z}) = \{x \in \mathbb{Z}^n : Q(x) = m\}$ of integral solutions.

In particular, we are interested in the asymptotic of the number $N(T) := \#\{x \in V_m(\mathbb{Z}) : \|x\| < T\}$ as $T \to \infty$, where $\|\cdot\|$ is a fixed norm on $\mathbb{R}^n$.

The answer to this question depends quite heavily on the geometry of the ambient space V_m. We suppose that the variety V_m is homogeneous, i.e., there exist a connected semisimple real algebraic group G defined over $\mathbb{Q}$ and a $\mathbb{Q}$-rational representation $\iota : G \to \mathrm{SL}_n$ such that $V_m = v_0.\iota(G)$ for some non-zero $v_0 \in \mathbb{Q}^n$.

Let $\Gamma < G(\mathbb{Q})$ be an arithmetic subgroup preserving $V_m(\mathbb{Z})$. By a theorem of Borel and Harish-Chandra [3], the co-volume of Γ in G is finite and there are only finitely many Γ-orbits in $V_m(\mathbb{Z})$. Hence understanding the asymptotic of $N(T)$ is reduced to the orbital counting problem on $\#(v_0\Gamma \cap B_T)$ for $B_T = \{x \in V_m : \|x\| < T\}$ and $v_0 \in V_m(\mathbb{Z})$.

Theorem 1.1. *Set H to be the stabilizer subgroup of v_0 in G. Suppose that H is either a symmetric subgroup or a maximal $\mathbb{Q}$-subgroup of G. If the*

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volume of $(H \cap \Gamma)\backslash H$ is finite, i.e., if $H \cap \Gamma$ is a lattice in H, we have

$$\#(v_0\Gamma \cap B_T) \sim \frac{\mathrm{vol}_H(H \cap \Gamma\backslash H)}{\mathrm{vol}_G(\Gamma\backslash G)} \mathrm{vol}_{H\backslash G}(B_T)$$

where the volumes on H, G and $v_0G \simeq H\backslash G$ are computed with respect to invariant measures chosen compatibly; that is, $d\,\mathrm{vol}_G = d\,\mathrm{vol}_H \times d\,\mathrm{vol}_{H\backslash G}$ locally.

This theorem was first proved by Duke, Rudnick, Sarnak [10] when H is symmetric and Eskin and McMullen gave a simplified proof in [11]. When H is a maximal $\mathbb{Q}$-subgroup, it is proved by Eskin, Mozes and Shah in [12].

As apparent from the main term of the asymptotic, it is crucial to assume $\mathrm{vol}(H \cap \Gamma\backslash H) < \infty$ in Theorem 1.1. The main aim of this paper is to break this barrier; to investigate the counting problem in the case when $\mathrm{vol}(H \cap \Gamma\backslash H) = \infty$.

We focus on the case when Q is a quadratic form of signature $(n-1, 1)$ with $n \geq 3$ and G is the special orthogonal group of Q. In this situation, the case of $\mathrm{vol}(H \cap \Gamma\backslash H) = \infty$ for $H = \mathrm{Stab}_G(v_0)$ arises only when $n = 3$ and $Q(v_0) = m > 0$, that is, when the variety $V_m = \{x \in \mathbb{R}^3 : Q(x) = m\}$ is a one-sheeted hyperboloid. To prove this claim, note first that if H is a non-compact simple Lie group, then any closed $\Gamma\backslash\Gamma H$ in $\Gamma\backslash G$ must be of finite volume by Dani [6] and Margulis [15] (see also [18]). Any non-compact stabilizer H of $v_0 \in \mathbb{R}^n$ in G is either locally isomorphic to $\mathrm{SO}(n-2, 1)$ (which is a simple Lie group except for $n = 3$) or a compact extension of a horospherical subgroup. Since any orbit of a horospherical subgroup is either compact or dense in $\Gamma\backslash G$ (cf. [8]), it follows that the case of $\mathrm{vol}(H \cap \Gamma\backslash H) = \infty$ arises only when $H \simeq \mathrm{SO}(1, 1)$; hence $n = 3$ and $Q(v_0) > 0$.

In the next subsection, we state our main theorem in a greater generality, not necessarily in the arithmetic situation.

1.2. **Counting integral points on a one-sheeted hyperboloid.** Let $Q(x_1, x_2, x_3)$ be an real quadratic form of signature $(2, 1)$. Denote by G the identity component of the special orthogonal group $\mathrm{SO}_Q(\mathbb{R})$. Let $\Gamma < G$ be a lattice and $v_0 \in \mathbb{R}^3$ be such that $Q(v_0) > 0$ and the orbit $v_0\Gamma$ is discrete. As before, we fix a norm $\|\cdot\|$ on $\mathbb{R}^3$ and set $B_T := \{x \in v_0G : \|w\| < T\}$.

To present our theorem with a best possible error term, we consider the following smoothed counting function: fixing a non-negative function $\psi \in C_c^\infty(G)$ with integral one, let

$$\tilde{N}_T := \sum_{v \in v_0\Gamma} (\chi_{B_T} * \psi)(v)$$

where $\chi_{B_T} * \psi(x) = \int_G \chi_{B_T}(xg)\psi(g)\, dg$, $x \in v_0G$, is the convolution of the characteristic function of B_T and ψ. Note that $\tilde{N}_T \asymp \#(v_0\Gamma \cap B_T)$ in the sense that their ratio is in between two uniform constants for all $T > 1$.

Denoting by $H \simeq \mathrm{SO}(1, 1)^\circ$ the one-dimensional stabilizer subgroup of v_0 in G, note that $\mathrm{vol}(H \cap \Gamma\backslash H) < \infty$ if and only if $H \cap \Gamma$ is infinite.

In order to state our theorem, we write H as a one-parameter subgroup $\{h(s) : s \in \mathbb{R}\}$ so that the Lebesgue measure ds defines a Haar measure on H: $\int_{-\log T}^{\log T} ds = \mathrm{vol}_H(\{h(s) : |s| < \log T\})$.

Theorem 1.2. *If the volume of $(H \cap \Gamma)\backslash H$ is infinite, we have the following:*

(1) *As $T \to \infty$,*
$$N_T \sim \frac{\int_{-\log T}^{\log T} ds}{\mathrm{vol}_G(\Gamma\backslash G)} \mathrm{vol}_{H\backslash G}(B_T)$$
where $d\,\mathrm{vol}_G = ds \times d\,\mathrm{vol}_{H\backslash G}$ locally.

(2) *for $T \gg 1$,*
$$\tilde{N}_T = c \cdot T \log T + O(T)$$
where $c = \lim_{T\to\infty} \frac{2\,\mathrm{vol}_{H\backslash G}(B_T)}{T\,\mathrm{vol}_G(\Gamma\backslash G)}$.

We note that when $\mathrm{vol}(H \cap \Gamma\backslash H) < \infty$, $\tilde{N}_T = c \cdot T + O(T^\alpha)$ for $0 < \alpha < 1$ is obtained in [10]. We believe, as suggested by Z. Rudnick to us, that $\tilde{N}_T = c \cdot T \log T + c' \cdot T + O(T^\alpha)$ for some $c' > 0$ and $0 < \alpha < 1$ and hence the order of the second term for $\tilde{N}_T$ cannot be improved.

Theorem 1.2 can be generalized to the orbital counting for more general representations of $\mathrm{SL}_2(\mathbb{R})$ (see section 6).

Remark 1.3. In the case when $Q = x_1^2 + x_2^2 - d^2x_3^2$ for $d \in \mathbb{Z}$, $v_0 = (1, 0, 0)$, and $\Gamma = \mathrm{SO}_Q(\mathbb{Z})$, it was pointed out in [10] that an elementary number theoretic computation of [17] leads to the asymptotic
$$\#\{(x_1, x_2, x_3) \in v_0\Gamma : \sqrt{x_1^2 + x_2^2 + d^2x_3^2} < T\} = c \cdot T \log T + O(T \log(\log T)).$$
However this deduction seems to work only for this very special case; for instance, we are not aware of any other approach than ours which can deal with non-arithmetic situtations.

1.3. **Arithmetic case and Integral binary quadratic forms.** In the arithmetic case, Theorem 1.2 together with Theorem 1.1 implies the following:

Corollary 1.4. *Let $Q(x_1, x_2, x_3)$ be an integral quadratic form with signature $(2, 1)$. Suppose that for some $v_0 \in \mathbb{Z}^3$ with $Q(v_0) > 0$, the stabilizer subgroup of v_0 is isotropic over $\mathbb{Q}$.*

Then there exists $c = c(\|\cdot\|) > 0$ such that as $T \to \infty$,
$$\#\{x \in \mathbb{Z}^3 : Q(x) = Q(v_0), \|x\| < T\} \sim c \cdot T \log T.$$

For a binary quadratic form $q(x, y) = ax^2 + bxy + cy^2$, its discriminant $\mathrm{disc}(q)$ is defined to be $b^2 - 4ac$. The group $\mathrm{SL}_2(\mathbb{R})$ acts on the space of binary quadratic forms by $(g.q)(x, y) = q(g^{-1}(x, y))$ and preserves the discriminant. For $d \in \mathbb{Z}$, denote by $\mathcal{B}_d(\mathbb{Z})$ the space of integral binary quadratic forms with discriminant d. Note that $\mathcal{B}_d(\mathbb{Z}) \neq \emptyset$ if and only if d congruent to 0 or 1 mod 4. Now d is a square *if and only if* the stabilizer of every $q \in \mathcal{B}_d(\mathbb{Z})$

in $\mathrm{SL}_2(\mathbb{Z})$ is infinite *if and only if* every $q \in \mathcal{B}_d(\mathbb{Z})$ is decomposable over $\mathbb{Z}$. (cf. [4]).

Therefore Corollary 1.4 implies the following:

Theorem 1.5. . *For any non-zero square $d \in \mathbb{Z}$, there exists $c_0 > 0$ such that*

$$\#\{q \in \mathcal{B}_d(\mathbb{Z}) : \mathrm{disc}(q) = d, \|q\| < T\} \sim c_0 \cdot T \log T$$

where $\|ax^2 + bxy + cy^2\| = \|(a, b, c)\|$.

1.4. **Orthogonal translates of a divergent geodesic.** Let $G = \mathrm{SL}_2(\mathbb{R})$ and Γ be a non-uniform lattice in G. For $s \in \mathbb{R}$, define

$$h(s) = \begin{bmatrix} \cosh(s/2) & \sinh(s/2) \\ \sinh(s/2) & \cosh(s/2) \end{bmatrix}, \ a(s) = \begin{bmatrix} e^{s/2} & 0 \\ 0 & e^{-s/2} \end{bmatrix} \tag{1.1}$$

and set $H = \{h(s) : s \in \mathbb{R}\}$.

In the case when the orbit $\Gamma\backslash\Gamma H$ is closed and of finite length, the limiting distribution of the translates $\Gamma\backslash\Gamma Ha(T)$ as $T \to \infty$ is described by the unique G-invariant probability measure $d\mu(g) = dg$ on $\Gamma\backslash G$ [10], that is, if s_0 is the period of $\Gamma \cap H\backslash H$, then for any $\psi \in C_c(\Gamma\backslash G)$,

$$\lim_{T\to\pm\infty} \frac{1}{s_0} \int_{s=0}^{s_0} \psi(h(s)a(T))ds = \int_{\Gamma\backslash G} \psi \, dg.$$

Similarly, understanding the limit of the translates $\Gamma\backslash\Gamma Ha(T)$ when $\Gamma\backslash\Gamma H$ is divergent (and of infinite length) is the main new ingredient in our proofs of Theorem 1.2.

Theorem 1.6. *Let $x_0 \in \Gamma\backslash G$ and suppose that $x_0 h(s)$ diverges as $s \to +\infty$, that is, $x_0 h(s)$ leaves every compact subset for all sufficiently large $s \gg 1$. For a given compact subset $\mathcal{K} \subset \Gamma\backslash G$, there exist $c = c(\mathcal{K}) > 0$ and $M = M(\mathcal{K}) > 0$ such that for any $\psi \in C^\infty(\Gamma\backslash G)$ with support in $\mathcal{K}$, we have, as $|T| \to \infty$,*

$$\int_0^\infty \psi(x_0 h(s)a(T))ds = \int_0^{T+M} \psi(x_0 h(s)a(T))ds = |T| \int \psi \, d\mu + O(1)$$

where the implied constant depends only on $\mathcal{K}$ and a Sobolev norm of ψ.

Remark 1.7. Consider the hyperbolic plane $\mathbb{H}^2$. A parabolic fixed point for Γ is a point in the geometric boundary $\partial_\infty(\mathbb{H}^2)$ fixed by a parabolic element of Γ. If $\mathcal{F} \subset \mathbb{H}^2$ is a finite sided Dirichlet region for Γ, then the parabolic fixed points of Γ are precisely the Γ-orbits of vertices of $\overline{\mathcal{F}}$ lying in $\partial_\infty(\mathbb{H}^2)$. Let $\pi : G \to \mathbb{H}^2$ denote the orbit map $g \mapsto g(i)$. For $x_0 = \Gamma g_0 \in \Gamma\backslash G$, the image $\pi(g_0 H)$ is a geodesic in $\mathbb{H}^2$ with two endpoints $g_0 H(+\infty) := \lim_{s\to\infty} \pi(g_0 h(s))$ and $g_0 H(-\infty) := \lim_{s\to-\infty} \pi(g_0 h(s))$ in $\partial_\infty(\mathbb{H}^2)$. We remark that $x_0 h(s)$ diverges as $s \to +\infty$ (resp. $s \to -\infty$) if and only if $g_0 H(+\infty)$ (resp. $g_0 H(-\infty)$) is a parabolic fixed point for Γ (cf. Theorem 2.1).

Corollary 1.8. *Suppose that x_0H is closed and non-compact. For any $\psi \in C_c(\Gamma\backslash G)$,*

$$\lim_{T\to\pm\infty} \frac{1}{2|T|}\int_{-\infty}^{\infty} \psi(x_0h(s)a(T))\,ds = \int_{\Gamma\backslash G} \psi\,d\mu.$$

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2. Structure of cusps in $\Gamma\backslash G$ and divergent trajectory

Let $G = \mathrm{SL}_2(\mathbb{R})$ and Γ be a non-uniform lattice in G. We will keep the notation for $h(s)$ and $a(s)$ from (1.1) in the introduction. Let

$$N = \{\left(\begin{smallmatrix} 1 & s \\ 0 & 1\end{smallmatrix}\right) : s \in \mathbb{R}\} \quad \text{and} \quad U = wNw^{-1}$$

where $w = \left[\begin{smallmatrix} \cos(\pi/4) & \sin(\pi/4) \\ -\sin(\pi/4) & \cos(\pi/4)\end{smallmatrix}\right]$. Note that $h(s) = wa(s)w^{-1}$ for all $s \in \mathbb{R}$. For $\eta > 0$, let

$$H_\eta = \{h(s) : s/2 > -\log\eta\}.$$

Let $K = \mathrm{SO}(2) = \{g \in G : gg^t = I\}$. Then the multiplication map $U\times H\times K \to G$: $(u,h,k)\mapsto uhk$ is a diffeomorphism.

The following classical result may be found at [13, Thm. 0.6] or [9]:

Theorem 2.1. *There exists a finite set $\Sigma \subset G$ such that the following holds:*

1. *$\Gamma\backslash\Gamma\sigma U$ is compact for every $\sigma\in\Sigma$.*
2. *For any $\eta > 0$, the set*
$$\mathcal{K}_\eta := \Gamma\backslash G \smallsetminus \bigcup_{\sigma\in\Sigma} \Gamma\backslash\Gamma\sigma U H_\eta K$$
is compact; and any compact subset of $\Gamma\backslash G$ is contained in $\mathcal{K}_\eta$ for some $\eta > 0$.
3. *There exists $\eta_0 > 0$ such that for $i = 1, 2$, if $\sigma_i \in \Sigma$, $u_i \in U$, $h_i \in H_{\eta_0}$, and $\Gamma\sigma_1u_1h_1k_1 = \Gamma\sigma_2u_2h_2k_2$, then $\sigma_1 = \sigma_2$, $k_1 = \pm k_2$ and $h_1 = h_2$.*

Consider the standard representation of $G = \mathrm{SL}_2(\mathbb{R})$ on $\mathbb{R}^2$: $((v_1, v_2), g) \mapsto (v_1, v_2)g$. Let $\|\cdot\|$ denote the Euclidean norm on $\mathbb{R}^2$. Let

$$p = (0,1)w^{-1} = (-\sin(\pi/4), \cos(\pi/4)).$$

Then $pU = p$, and $ph(s) = (0,1)a(s)w^{-1} = e^{-s/2}p$ for all $s \in \mathbb{R}$. Also

$$g \in UH_\eta K \Leftrightarrow \|pg\| < \eta. \tag{2.1}$$

Proposition 2.2 (Dani [7])**.** *Let $x_0 \in \Gamma\backslash G$ be such that the trajectory $\{x_0h(s) : s \geq 0\}$ is divergent. Then there exist $\sigma_0 \in \pm I\Sigma$, $s_0 \in \mathbb{R}$ and $u \in U$ such that $x_0 = \Gamma\sigma_0 uh(s_0)$.*

Proof. By Theorem 2.1, there exists $s_1 > 0$ and $\sigma \in \Sigma$ such that $x_0h(s) = \Gamma\sigma UH_{\eta_0/2}K$ for all $s \geq s_1$. Let $g_1 \in UH_{\eta_0/2}K$ be such that $x_0h(s_1) = \Gamma\sigma g_1$. We claim that $pg_1 \in \mathbb{R}p$. If not, then $\|ph(s)\| \to \infty$ as $s \to \infty$, and hence

there exists $s > 0$ such that $\eta_0/2\|pg_1h(s)\| < \eta_0$. By (2.1), $g_1h(s) \in uhk$ for some $u \in U$, $h \in H_{\eta_0}$ and $k \in K$. Therefore

$$\Gamma\sigma uhk = \Gamma\sigma g_1 h(s) = x_0 h(s_1 + s) \in \Gamma\sigma U H_{\eta_0/2} K.$$

By Theorem 2.1(3), we have that $h \in H_{\eta_0/2}$. But then $\|pg_1h(s)\| = \|puhk\| < \eta_0/2$, a contradiction. Therefore our claim that $pg_1 \in \mathbb{R}p$ is valid. Hence $g_1 = u_1h(s)\{\pm I\}$ for some $u_1 \in U$ and $s/2 \geq -\log(\eta_0/2)$. Thus $x_0h(s_1) = \Gamma\sigma u_1 h(s)\{\pm I\}$, and hence $x_0 = \Gamma\sigma_0 u_1 h(s - s_1)$, where $\sigma_0 = \pm I\sigma$. □

Proposition 2.3. *Let $x_0 \in \Gamma\backslash G$ be such that the trajectory $\{x_0h(s) : s \geq 0\}$ is divergent. Let $\mathcal{K} \subset \Gamma\backslash G$ be a compact subset. There exists $M_1 = M_1(\mathcal{K}) > 0$ such that*

$$x_0 h(s) a(T) \notin \mathcal{K}$$

for any $T \in \mathbb{R}$ and $s > 0$ satisfying $s > |T| + M_1$. In particular, for any $f \in C(\Gamma\backslash G)$ with support inside $\mathcal{K}$,

$$\int_0^\infty f(x_0h(s)a(T))\, ds = \int_0^{|T|+M_1} f(x_0h(s)a(T))\, ds.$$

Proof. By Proposition 2.2, $x_0 = \Gamma\sigma_0 uh(s_0)$ for some $\sigma_0 \in \pm\Sigma, u \in U, s_0 \in \mathbb{R}$. By Theorem 2.1(2), let $\eta > 0$ be such that $\mathcal{K} \subset \mathcal{K}_\eta$. Let $M_1 = -s_0 - 2\log(\eta)$. Since $s - |T| > -s_0 - 2\log\eta$, we have

$$\begin{aligned} \|puh(s_0)h(s)a(T)\| &= \|ph(s+s_0)a(T)\| \\ &= e^{-(s+s_0)/2}\|pa(T)\| \\ &< e^{-(s+s_0)/2}e^{|T|/2} \\ &= e^{-(s+s_0-|T|)/2} < \eta. \end{aligned} \tag{2.2}$$

Therefore by (2.1), $uh(s_0)h(s)a(T) \in UH_\eta K$, and hence

$$x_0h(s)a(T) \in \Gamma\sigma_0 U H_\eta K \subset \Gamma\backslash G \smallsetminus K_\eta.$$

□

3. Uniform mixing on compact sets

Let $G = \mathrm{SL}_2(\mathbb{R})$ and $\Gamma < G$ be a lattice. Let μ denote the G-invariant probability measure on $\Gamma\backslash G$. For an orthonormal basis X_1, X_2, X_3 of $\mathfrak{sl}(2,\mathbb{R})$ with respect to an Ad-invariant scalar product, and $\psi \in C^\infty(\Gamma\backslash G)$, we consider the Sobolev norm

$$\mathcal{S}_m(\psi) = \max\{\|X_{i_1}\cdots X_{i_j}(\psi)\|_2 : 1 \leq i_j \leq 3, 0 \leq j \leq m\}.$$

The well-known spectral gap property for $L^2(\Gamma\backslash G)$ says that the trivial representation is isolated (see [2, Lemma 3]) in the Fell topology of the unitary dual of G. It follows that there exist $\theta > 0$ and $c > 0$ such that for any $\psi_1, \psi_2 \in C^\infty(\Gamma\backslash G)$ with $\int \psi_i d\mu = 0$, $\mathcal{S}_1(\psi_i) < \infty$ and for any $T > 0$,

$$|\langle a(T)\psi_1, \psi_2\rangle| \leq ce^{-\theta|T|}\mathcal{S}_1(\psi_1)\mathcal{S}_1(\psi_2) \tag{3.1}$$

(cf. [5], [19])

Write $\mathcal{O}_\epsilon = \{g \in G : \|g - I\|_\infty \le \epsilon\}$. For a compact subset $\mathcal{K} \subset \Gamma\backslash G$, let $0 < \epsilon_0(\mathcal{K}) \le 1$ be the injectivity radius of $\mathcal{K}$, that is, $\epsilon_0(\mathcal{K})$ is the supremum of $0 < \epsilon \le 1$ such that the multiplication map $\mathcal{K} \times \mathcal{O}_\epsilon \to \Gamma\backslash G$ is injective.

For $s \in \mathbb{R}$, let

$$n_+(s) = \left(\begin{smallmatrix} 1 & 0 \\ s & 1 \end{smallmatrix}\right) \quad \text{and} \quad n_-(s) = \left(\begin{smallmatrix} 1 & s \\ 0 & 1 \end{smallmatrix}\right).$$

Theorem 3.1. *Let $\mathcal{K} \subset \Gamma\backslash G$ be a compact subset and $\eta > 0$. There exists $c = c(\mathcal{K}) > 0$ such that for any $\psi \in C^\infty(\Gamma\backslash G)$ with support in $\mathcal{K}$, for any $|T| \ge 1$, $x \in \mathcal{K}$, and $0 < r_0 < \epsilon_0(\mathcal{K})$, we have*

$$\Big|\int_0^{r_0} \psi(xn_\nu(s)a(T))\, ds - r_0 \int \psi\, d\mu\Big| \le c(\mathcal{S}_3(\psi) + 1)e^{-\theta_0|T|} \tag{3.2}$$

for some θ_0 depending only on the spectral gap for $L^2(\Gamma\backslash G)$. Here and in what follows, the sign $\nu = +$ if $T > 0$ and $\nu = -$ if $T < 0$.

Proof. Consider the case when $T > 0$ and hence $\nu = +$. The other case can be proved similarly.

Let $\epsilon > 0$. Fix a non-negative function $\rho_\epsilon \in C_c^\infty(N^+)$ which is 1 on $n_+[0, r_0]$ and 0 outside $n_+[-\epsilon, r_0 + \epsilon]$. Let $N^\pm = \{n_\pm(s) : s \in \mathbb{R}\}$ and $W_\epsilon := AN^- \cap \mathcal{O}_\epsilon$. Let μ_0 denote the right invariant measure on AN^- such that $d\mu_0 \otimes dn = d\mu$. We choose a non-negative function $\phi_\epsilon \in C^\infty(AN^-)$ supported inside W_ϵ and $\int \phi_\epsilon d\mu_0 = 1$.

If we a consider a function $\tau_{x,\epsilon}$ on $\Gamma\backslash G$ which is defined to be $\tau_{x,\epsilon}(y) := \rho_\epsilon(n_+(s))\phi_\epsilon(w) \in C^\infty(\Gamma\backslash G)$ if $y = xn_+(s)w \in x\,\mathrm{supp}(\rho_\epsilon)W_\epsilon$ and 0 otherwise, then $\mathcal{S}_1(\tau_{x,\epsilon}) \ll \epsilon^{-3}$ where the implied constant is independent of x and

$$\begin{aligned} \langle a(T)\psi, \tau_{x,\epsilon}\rangle &= \int_{y\in\Gamma\backslash G} \psi(ya(T))\tau_{x,\epsilon}(y)d\mu(y) \\ &= \int_{u\in W_\epsilon, s\in\mathbb{R}} \psi(xn_+(s)wa(T))\phi_\epsilon(w)\rho_\epsilon(n_+(s))d\mu_0(w)ds. \end{aligned} \tag{3.3}$$

As $\mathcal{K}$ is compact, the C^1-norm of f supported inside $\mathcal{K}$ is bounded above by a uniform multiple of $\mathcal{S}_3(\psi)$ (cf. [1, Thm 2.20]) and hence for some $c_1 > 0$,

$$\max\{\|\psi\|_\infty, C_\psi\} < c_1\mathcal{S}_3(\psi) \tag{3.4}$$

where C_ψ is the Lipschitz constant of ψ.

Since for all $T > 0$, $W_\epsilon a(T) \subset a(T)\mathcal{O}_{2\epsilon}$, we have for all $w \in W_\epsilon$ and $T \gg 1$,

$$\Big|\psi(xn_+(s)wa(T)) - \psi(xn_+(s)a(T))\Big| \le 2c_1\mathcal{S}_3(\psi)\epsilon.$$

Hence by (3.3),

$$\begin{aligned}
&\Big|\langle a(T)\psi,\tau_{x,\epsilon}\rangle - \int_{s\in\mathbb{R}} \psi(xn_+(s)a(T))\rho_\epsilon(n_+(s))ds\Big| \\
&= \Big|\int_{w,s} \psi(xn_+(s)wa(T))\phi_\epsilon(w)\rho_\epsilon(n_+(s))d\mu_0(w)ds \\
&\qquad\qquad - \int_{s\in\mathbb{R}} \psi(xn_+(s)a(T))\rho_\epsilon(n_+(s))ds\Big| \\
&\ll 2c_1\mathcal{S}_3(\psi)\epsilon\|\rho_\epsilon\|_1 \le 2c_1\mathcal{S}_3(\psi)\epsilon(r_0+2\epsilon).
\end{aligned}$$

Since

$$\Big|\langle a(T)\psi,\tau_{x,\epsilon}\rangle - \int \psi d\mu\cdot\|\rho_\epsilon\|_1\Big| \ll e^{-\theta T}\epsilon^{-3}\mathcal{S}_1(\psi),$$

we deduce

$$\begin{aligned}
&\Big|\int_0^{r_0} \psi(xn_+(s)a(T))ds - r_0\int\psi d\mu\Big| \\
&\le \Big|\int_{s\in\mathbb{R}} \psi(xn_+(s)a(T))\rho_\epsilon(n_+(s))ds - r_0\int\psi d\mu\|\rho_\epsilon\|_1\Big| + 4c_1\epsilon\mathcal{S}_3(\psi) \\
&\le \Big|\langle a(T)\psi,\tau_{x,\epsilon}\rangle - r_0\int\psi d\mu\cdot\|\rho_\epsilon\|_1\Big| + 6c_1\epsilon\mathcal{S}_3(\psi) \\
&\le 6c_1\epsilon\mathcal{S}_3(\psi) + c\cdot e^{-\theta T}\epsilon^{-3}\mathcal{S}_3(\psi)
\end{aligned}$$

for some $c>0$. Hence for $\epsilon = e^{-\theta T/4}$ and some $c_2>0$,

$$\Big|\int_0^{r_0} \psi(xn_+(s)a(T))ds - r_0\int\psi d\mu\Big| \le c_2(\mathcal{S}_3(\psi)+1)e^{-\theta T/4}.$$

□

4. Translates of divergent orbits

Let $x_0\in\Gamma\backslash G$ be such that $x_0h(s)$ diverge as $s\to\infty$.

Theorem 4.1. *For any $|T|>1$ and any $\psi\in C_c^\infty(\Gamma\backslash G)$*

$$\int_0^{|T|}\psi(x_0h(s)a(T))\,ds = |T|\int\psi\,d\mu + O(1)\mathcal{S}_3(\psi).$$

Proof. Let $R_0 = -\log\eta_0$. Due to Proposition 2.2, replacing x_0 by another point in x_0H, we may assume that $x_0 = \Gamma\sigma_0h(R_0)$. For any $S>0$, $\|ph(R_0)h(S)a(S)\|\in[\eta_0/\sqrt{2},\eta_0]$. Hence $x_0h(R_0)h(S)a(S)\in K_{\eta_0/\sqrt{2}}$.

Let r_0 be the injectivity radius of $K_{\eta_0/\sqrt{2}}$, that is, $r_0=\epsilon_0(K_{\eta_0/\sqrt{2}})$. Let $S_0=0$, and choose S_i such that $r_0e^{-S_i}\le\delta_i := S_{i+1}-S_i\le 2r_0e^{-S_i}$ for each i. We will choose $S_i=\log(2r_0i+1)$ for each i. Then $x_0h(S_i)a(S_i)\in K_{\eta_0/\sqrt{2}}$. We put $R_i=T-S_i$.

We will express $x_0h([S_i,S_{i+1}])a(T) = x_ih^{a(S_i)}([0,\delta_i])a(R_i)$, where $x_i = x_0h(S_i)a(S_i)$ and $h^{a(S_i)}(s)=a(-S_i)h(s)a(S_i)=n(e^{S_i}s/2)w_i(s)$, and $|w_i(s)| = O(e^{-2S_i})$. Note that $r_0/2\le e^{S_i}\delta_i/2\le r_0$.

By Theorem 3.1, we have

$$\int_0^{r_0} \psi(x_i n(s) a(R_i)) ds - r_0 \int \psi d\mu = \mathcal{S}_3(\psi) \cdot O(e^{-\theta_0 R_i})$$

and hence

$$\int_{S_i}^{S_{i+1}} \psi(x_0 h(s) a(T)) ds = \frac{\delta_i}{r_0} \int_0^{r_0} \psi(x_i n(s) a(R_i)) ds + \mathcal{S}_3(\psi) \cdot O(e^{-2S_i} \delta_i).$$

Let $k = k(T)$ be such that $S_k \le T < S_k + r_0 e^{-S_k}$. Therefore, since $\delta_i r_0^{-1} \le 2e^{-S_i}$,

$$\int_0^T \psi(xh(s)a(T)) ds = \sum_{i=0}^{k-1} \int_{S_i}^{S_{i+1}} \psi(xh(s)a(T)) ds + O(e^{-S_k})$$

$$= \sum_{i=0}^{k-1} \delta_i \frac{1}{r_0} \int_0^{r_0} \psi(x_i n(s) a(T)) ds + \mathcal{S}_3(\psi) \cdot O(e^{-2S_i} \delta_i) + O(1)$$

$$= \sum_{i=0}^{k-1} \delta_i \mu(\psi) + \sum_{i=0}^{k-1} \delta_i r_0^{-1} \mathcal{S}_3(\psi) \cdot O(e^{-\theta_0 R_i}) + \mathcal{S}_3(\psi) \cdot O(e^{-2S_i} \delta_i) + O(1)$$

$$= T\mu(\psi) + O(\sum_{i=1}^{k-1} e^{-S_i} e^{-\theta_0 R_i} + \sum_{i=1}^{k} e^{-3S_i}) \mathcal{S}_3(\psi) + O(1)$$

$$= T\mu(\psi) + O(e^{-\theta_0 T} \sum_{i=0}^{k-1} e^{(1-\theta_0) S_i} + \sum_{i=0}^{k-1} e^{-3S_i}) \mathcal{S}_3(\psi) + O(1).$$

Since $S_i = \log(2r_0 i + 1)$, $0 < T - S_k < 2e^{-T}$ implies that $k < \frac{e^T - 1}{2r_0} < k+1$, and hence

$$\sum_{i=0}^{k-1} e^{-3S_i} \ll \sum_{i=1}^{k-1} \frac{1}{(2r_0 i + 1)^3} = O(k^{-2} + 1) = O(e^{-2T} + 1) < \infty$$

and

$$\sum_{i=0}^{k-1} e^{(1-\theta_0) S_i} \ll \int_0^{e^T} \frac{1}{(2r_0 x + 1)^{1-\theta_0}} dx = O(e^{\theta_0 T}).$$

Hence

$$e^{-\theta_0 T} \sum_{i=0}^{k-1} e^{(1-\theta_0) S_i} + \sum_{i=0}^{k-1} e^{-3S_i} = O(1).$$

Therefore

$$\int_0^T \psi(xh(s)a(T)) ds = T\mu(\psi) + O(1) \mathcal{S}_3(\psi).$$

□

Theorem 1.6 follows from the following:

Theorem 4.2. *Let $x_0h(s)$ diverge as $s \to \infty$. For a given compact subset $\mathcal{K} \subset \Gamma\backslash G$, and $\psi \in C^\infty(\Gamma\backslash G)$ with support in $\mathcal{K}$, we have*

$$\int_0^\infty \psi(x_0h(s)a(T))ds = |T| \cdot \int \psi \, d\mu + O(1)\mathcal{S}_3(\psi).$$

Proof. Since $x_0h(s)$ diverges as $s \to \infty$, by Proposition 2.3, there exists $M_1 = M_1(\mathcal{K}) > 0$ such that

$$\int_0^\infty \psi(x_0h(s)a(T))ds = \int_0^{|T|+M_1} \psi(x_0h(s)a(T))ds$$

$$= (|T| + M_1) \int \psi \, d\mu + O(1)\mathcal{S}_3(\psi)$$

$$= |T| \int \psi \, d\mu + O(1)\mathcal{S}_3(\psi).$$

□

By a similar argument, we also deduce the following:

Corollary 4.3. *If $x_0h(s)$ diverges as $s \to -\infty$, then*

$$\int_{-\infty}^0 \psi(x_0h(s)a(T))ds = |T| \int \psi d\mu + O(1)\mathcal{S}_3(\psi).$$

Lemma 4.4. *If $x_0h(\mathbb{R})$ is closed and non-compact, then $x_0h(s)$ diverges as $s \to \pm\infty$.*

Proof. We use a well-known fact that for a closed subgroup H of a locally compact second countable group G and a discrete subgroup Γ of G, if ΓH is closed in G, then the canonical projection map $H \cap \Gamma\backslash H \to \Gamma\backslash G$ is a proper map (cf. [16]). Since $x_0h(\mathbb{R})$ is non-compact and $h(\mathbb{R})$ is one-dimensional with no non-trivial finite subgroups, the stabilizer of x_0 in $h(\mathbb{R})$ is trivial. Therefore the map $h(\mathbb{R}) \to \Gamma\backslash G$ given by $h \to x_0h$ is a proper injective map. This implies that $x_0h(s)$ diverges as $s \to \pm\infty$. □

Proof of Corollary 1.8. As the set $C_c^\infty(\Gamma\backslash G)$ is dense in $C_c(\Gamma\backslash G)$, the claim follows from Lemma 4.4, Theorem 4.1, and Corollary 4.3. □

5. Counting: Proof of Theorem 1.2

Let Q be a real quadratic form in 3 variables of signature $(2,1)$ and Γ_0 a lattice in the identity component G_0 of $\mathrm{SO}_Q(\mathbb{R})$. We assume that $v_0\Gamma_0$ is discrete for some vector $v_0 \in \mathbb{R}^3$ with $Q(v_0) = d > 0$ and that the stabilizer H_0 of v_0 in G_0 is finite.

It suffices to prove Theorem 1.2 in the case when $Q = x^2 + y^2 - z^2$ and $v_0 = (\sqrt{d}, 0, 0)$ by the virtue of Witt's theorem.

Consider the spin double cover map $\iota : G := \mathrm{SL}_2(\mathbb{R}) \to G_0$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} \frac{a^2-b^2-c^2+d^2}{2} & ac-bd & \frac{a^2-b^2+c^2-d^2}{2} \\ ab-cd & bc+ad & ab+cd \\ \frac{a^2+b^2-c^2-d^2}{2} & ac+bd & \frac{a^2+b^2+c^2+d^2}{2} \end{pmatrix}.$$

For $s \in \mathbb{R}$, we set

$$h(s) = \begin{pmatrix} \cosh(s/2) & \sinh(s/2) \\ \sinh(s/2) & \cosh(s/2) \end{pmatrix}; \quad \text{and} \quad a(s) = \begin{pmatrix} e^{s/2} & 0 \\ 0 & e^{-s/2} \end{pmatrix}.$$

Recall that $H := \{h(s) : s \in \mathbb{R}\}$, $A := \{a(t) : t \in \mathbb{R}\}$ and $K_1 := \{k(\theta) : \theta \in [0, 2\pi]\}$, here K_1 is half of the circle group. Observing that

$$\iota(h(s)) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh s & \sinh s \\ 0 & \sinh s & \cosh s \end{pmatrix} \quad \text{and} \quad \iota(a(t)) = \begin{pmatrix} \cosh t & 0 & \sinh t \\ 0 & 1 & 0 \\ \sinh t & 0 & \cosh t \end{pmatrix},$$

the subgroup $\tilde{H} := \pm H$ is the stabilizer of v_0 in G. We have a generalized Cartan decomposition $G = \tilde{H}AK_1$ in the sense that every g is of the form hak for unique $h \in \tilde{H}, a \in A, k \in K_1$. And for $g = h(s)a(t)k$, $d\mu(g) = \sinh(t)dsdtdk$ defines a Haar measure on G, where $dk = (1/2\pi)dk(\theta)$, and ds, dt and $d\theta$ are Lebesgue measures. As $v_0G = \pm H\backslash G \simeq A \times K_1$, $\sinh(t)dtdk$ defines an invariant measure on v_0G. We consider the volume forms on G and v_0G with respect to these measures. Via the map ι, these define invariant measures on G_0 and v_0G_0 as well.

Denote by Γ the pre-image of Γ_0 under ι. Then $\mathrm{Stab}_\Gamma(v_0) = \tilde{H} \cap \Gamma = \{\pm I\}$.

For each $T > 1$, define a function on $\Gamma\backslash G$:

$$F_T(g) := \sum_{\gamma \in \pm I \backslash \Gamma} \chi_{B_T}(v_0\gamma g).$$

Proposition 5.1. *For any $\Psi \in C_c^\infty(\Gamma\backslash G)$,*

$$\langle F_T, \Psi\rangle = \frac{T \log T \mu(\Psi)}{\mathrm{vol}(\Gamma\backslash G)} \cdot 2\int_{K_1} \frac{1}{\|v^+k\|} dk + O(T)\mathcal{S}_3(\psi)$$

where $v^\pm = \frac{\sqrt{d}}{2}(e_1 \pm e_3)$. Here the implied constant depends only on $\mathcal{S}_3(\Psi)$ and the support of Ψ.

Proof. Then $v_0 = v^+ + v^-$ and $v_0a(t) = e^t v^+ + e^{-t}v^-$. Since $B_T = \{v_0a(t)k : \|v_0a(t)k\| < T\, t \in \mathbb{R},\, k \in K_1\}$, we have

$$\begin{aligned}
\langle F_T, \Psi\rangle &= \int_{\Gamma\backslash G} \sum_{\gamma \in \pm I\backslash\Gamma} \chi_{B_T}(v_0\gamma g)\Psi(g)d\mu(g) \\
&= \int_{k\in K_1} \int_{\|v_0a(t)k\|<T} \left(\int_{h(s) \in \pm I\backslash \tilde{H}} \Psi(h(s)a(t)k)ds \right) \sinh(t)dtdk \\
&= \int_{k\in K_1} \int_{\|v_0a(t)k\|<T} \left(\int_{s\in\mathbb{R}} \Psi(h(s)a(t)k)ds \right) \sinh(t)dtdk.
\end{aligned}$$

Since $v_0\Gamma$ is discrete and $H \cap \Gamma$ is trivial, it follows that $\Gamma\backslash\Gamma H$ is closed and non-compact in $\Gamma\backslash G$. Now fix any $k \in K_1$. Hence by Theorem 4.2 and

Lemma 4.4,

$$\begin{aligned}
&\int_{t\gg 1,\|v_0a(t)k\|<T}\left(\int_{s\in\mathbb{R}}\Psi(h(s)a(t)k)ds\right)\sinh(t)dt\\
&=\frac{1}{\mathrm{vol}(\Gamma\backslash G)}\int_{t\gg 1,e^t\|v^+k\|<T+O(1)}(2t\mu(\psi)+O(1)\mathcal{S}_3(\psi))(e^t/2+O(1))dt\\
&=\frac{T\log T\mu(\Psi)}{\mathrm{vol}(\Gamma\backslash G)\cdot\|v^+k\|}+O(T)\mathcal{S}_3(\psi).
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\int_{t\ll -1,\|v_0a(t)k\|<T}\left(\int_{s\in\mathbb{R}}\Psi(h(s)a(t)k)ds\right)\sinh(t)dt\\
&=\int_{t\gg 1,\|v_0a(-t)k\|<T}\left(\int_{s\in\mathbb{R}}\Psi(h(s)a(-t)k)ds\right)\sinh(t)dt\\
&=\frac{1}{\mathrm{vol}(\Gamma\backslash G)}\int_{t\gg 1,e^t\|v^-k\|<T+O(1)}(2t\mu(\psi)+O(1)\mathcal{S}_3(\psi))(e^t/2+O(1))dt\\
&=\frac{T\log T\mu(\Psi)}{\mathrm{vol}(\Gamma\backslash G)\|v^-k\|}+O(T)\mathcal{S}_3(\psi).
\end{aligned}$$

Since $v^-k(\pi)=-v^+$,

$$\int_{k\in K_1}\|v^-k\|^{-1}dk=\int_{k\in K_1}\|v^+k(\pi)k\|^{-1}dk=\int_{K_1}\|v^+k\|^{-1}dk.$$

The required formula can be deduced in a straightforward manner from this. □

Fix a non-negative function $\psi\in C_c^\infty(G)$ whose support injects to $\Gamma\backslash G$ and with integral $\int\psi(g)\,d\mu(g)=1$. Consider a function ξ_T on $\mathbb{R}^3$ defined by

$$\xi_T(x)=\int_{g\in G}\chi_{B_T}(xg)\psi(g)d\mu(g).$$

Then the sum $\sum_{\gamma\in\pm I\backslash\Gamma}\xi_T(v_0\gamma)$ is a smoothed over counting satisfying

$$\sum_{\gamma\in\pm I\backslash\Gamma}\xi_T(v_0\gamma)\asymp\#v_0\Gamma\cap B_T.$$

Theorem 5.2. *As $T\to\infty$,*

$$\sum_{\gamma\in\pm I\backslash\Gamma}\xi_T(v_0\gamma)=\frac{2T\log T}{\mathrm{vol}(\Gamma\backslash G)}\cdot\int_{k\in K_1}\frac{1}{\|v^+k\|}dk+O(T)\mathcal{S}_3(\psi).$$

Proof. It is not hard to verify that

$$\sum_{\gamma\in\pm I\backslash\Gamma}\xi_T(v_0\gamma)=\langle F_T,\Psi\rangle$$

where $\Psi(\Gamma g)=\sum_{\gamma\in\Gamma}\psi(\gamma g)$. Therefore the claim follows from Proposition 5.1. □

Theorem 5.3. *For $T \gg 1$, we have*

$$\#\{w \in v_0\Gamma : \|w\| < T\} = \frac{2T\log T}{\mathrm{vol}(\Gamma\backslash G)} \int_{K_1} \frac{1}{\|w^+k\|} dk (1 + (\log T)^{-\alpha})$$

where $\alpha = -1/5.5$.

Proof. Note that $F_T(I) = \#\{w \in v_0\Gamma : \|w\| < T\}$. For each $\epsilon > 0$, let $\mathcal{O}_\epsilon = \{g \in G : \|g - I\|_\infty \le \epsilon\}$. There exists $0 < \ell \le 1$ such that for all small $\epsilon > 0$,

$$\mathcal{O}_{\ell\epsilon} B_T \subset B_{(1+\epsilon)T}, \quad B_{(1-\epsilon)T} \subset \cap_{u \in \mathcal{O}_{\ell\epsilon}} u B_T. \tag{5.1}$$

Let ψ^ϵ be a non-negative smooth function on G supported in $\mathcal{O}_{\ell\epsilon}$ and with integral $\int \psi^\epsilon d\mu = 1$ and define $\Psi^\epsilon \in C_c^\infty(\Gamma\backslash G)$ by $\Psi^\epsilon(\Gamma g) := \sum_{\gamma\in\Gamma} \psi^\epsilon(\gamma g)$.

Using (5.1), we have

$$\langle F_{(1-\epsilon)T}, \Psi^\epsilon\rangle \le F_T(I) \le \langle F_{(1+\epsilon)T}, \Psi^\epsilon\rangle.$$

Therefore by Proposition 5.1

$$\begin{aligned}\langle F_{(1\pm\epsilon)T}, \Psi^\epsilon\rangle &= \frac{2T\log T}{\mathrm{vol}(\Gamma\backslash G)} \int_{K_1} \frac{1}{\|w^+k\|} dk + O(\epsilon T \log T) + O(\mathcal{S}_3(\Psi^\epsilon)T) \\ &= \frac{2T\log T}{\mathrm{vol}(\Gamma\backslash G)} \int_{K_1} \frac{1}{\|w^+k\|} dk (1 + (\log T)^{-1/5.5},\end{aligned}$$

where the last equality follows because $\mathcal{S}_3(\Psi^\epsilon) = O(\epsilon^{-4.5})$, and if we put $\epsilon = (\log T)^{-1/5.5}$ then

$$O(\mathcal{S}_3(\Psi^\epsilon)T) = O(\epsilon T \log T) = (T\log T)(\log T)^{-1/5.5}.$$

□

Proof of Theorem 1.2. The above computation in the proof of Proposition 5.1 also shows that

$$\mathrm{vol}(B_T) = \int_{k\in K_1} \int_{\|v_0 a(t) k\| < T} \sinh(t) dt dk = T \int_{k\in K} \frac{1}{\|v^+k\|} dk + O(\log T). \tag{5.2}$$

From Theorem 5.3, it follows that

$$F_T(I) = \frac{2\log T \,\mathrm{vol}(B_T)}{\mathrm{vol}(\Gamma\backslash G)} (1 + O(\log T)^{-\alpha})). \tag{5.3}$$

Since $F_T(I) = \#(v_0\Gamma \cap B_T)$, this completes the proof. □

6. Orbital counting for general representations of $\mathrm{SL}_2(\mathbb{R})$

Let $G = \mathrm{SL}_2(\mathbb{R})$ and Γ be a non-uniform lattice in G. For $s \in \mathbb{R}$, define

$$h(s) = \begin{bmatrix} \cosh(s/2) & \sinh(s/2) \\ \sinh(s/2) & \cosh(s/2) \end{bmatrix}, \; a(s) = \begin{bmatrix} e^{s/2} & 0 \\ 0 & e^{-s/2} \end{bmatrix}, \; k(\theta) = \begin{bmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{bmatrix}$$

Put $H = \{h(s) : s \in \mathbb{R}\}$, $A^+ = \{a(t) : t > 0\}$, and $K_1 = \{k(\theta) : \theta \in [0, 2\pi]\}$, here K_1 is half of the circle group. Put $w_0 = k(\pi)$. Then $\{\pm I\}\backslash G = HA^+K_1 \cup Hw_0A^+K_1$, $w_0^{-1}h(s)w_0 = h(-s)$ and $w_0^{-1}a(t)w_0 = a(-t)$.

Let V be any finite dimensional representation of G and $v_0 \in G$ be such that H is the stabilizer subgroup of v_0 in G, i.e., $H = G_{v_0}$ where $G_{v_0} = \{g \in G : v_0 g = v_0\}$. Assume that V is linearly spanned by $v_0 G$. Then if e^{mt} is the highest eigenvalue for $a(t)$-action on V, then $m \in \mathbb{N}$, and the G action factors through $\{\pm I\}\backslash G = \mathrm{PSL}_2(\mathbb{R}) \cong \mathrm{SO}(2,1)^0$.

For example, let V_m denote the $(2m+1)$-dimensional space of real homogeneous polynomials of degree $2m$ in two variables, and consider the standard right action of $g \in \mathrm{SL}(2,\mathbb{R})$ on $P(x,y) \in V_m$ by $(Pg)(x,y) = P((x,y)g)$, where $(x,y)\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] = (ax+cy, bx+dy)$. Let $v_0(x,y) = (x^2-y^2)^m$. Then $G_{v_0} = H\mathcal{W}$, where $\mathcal{W} = \{\pm I\}$ if m is odd and $\mathcal{W} = \{\pm I, \pm w_0\}$ if m is even. Moreover, $\{P \in V_m : Ph = P, \text{ for all } h \in H\} = \mathbb{R}P_0$. A general finite dimensional representation of G with a nonzero H-fixed vector is a direct sum of such irreducible representations, and v_0 is a sum of one nonzero H-fixed vector from each of the irreducible representations; we assume that V is a span of $v_0 G$.

Theorem 6.1. *Let V, v_0 and m be as above. Suppose that Γ is a lattice in G, $v_0\Gamma$ is discrete, and $\Gamma_{v_0} := \Gamma \cap G_{v_0}$ is finite. Let $\|\cdot\|$ be any norm on V, and $v_0^+ = \lim_{t\to\infty} v_0 a_t / \|v_0 a_t\|$. Let C be an open subset of $\{v \in V : \|v\| = 1\}$ such that $\Theta = \{\theta \in [0,2\pi] : v_0^+ k(\theta) \in \mathbb{R}C\}$ has positive Lebesgue measure, and $\{\theta \in [0,2\pi] : v_0^+ k(\theta) \in \mathbb{R}(\overline{C} \smallsetminus C)\}$ has zero Lebesgue measure. Then for $T \gg 1$,*

$$\begin{aligned} &\#(v_0\Gamma \cap [0,T]C) \\ &\quad = \frac{4(2\pi)^{-1}\int_\Theta \|v_0^+ k(\theta)\|^{-1/m}\, d\theta}{|\Gamma_{v_0}| \cdot \mathrm{vol}_G(\Gamma\backslash G)} \times \frac{\log T}{m} T^{1/m}(1 + (\log T)^{-\alpha}) \end{aligned} \tag{6.1}$$

where vol_G is given by the Haar integral $dg = \sinh(t) dt ds d\theta$ on G, where $g = h(s)a(t)k(\theta)$, and $\alpha = \frac{1}{5.5}$.

Moreover, if $C \subset V$ satisfies $\mathbb{R}\overline{C} \cap v_0^+ K_1 = \emptyset$, then $\#(v_0\Gamma \cap \mathbb{R}C) < \infty$.

Proof. The result can be deduced by the arguments as in the proof of Theorem 5.3; one may also use the basic ideas from [16] about using the highest weight. □

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